

Computational information for the logistic map at the chaos threshold

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Abstract

We study the logistic map $f(x) = \lambda x(1 - x)$ on the unit square at the chaos threshold. By using the methods of symbolic dynamics, the information content of an orbit of a dynamical system is defined as the *Algorithmic Information Content (AIC)* of a symbolic sequence. We give results for the behaviour of the AIC for the logistic map. Since the AIC is not a computable function we use, as approximation of the AIC, a notion of information content given by the length of the string after it has been compressed by a compression algorithm, and in particular we introduce a new compression algorithm called CASToRe. The information content is then used to characterise the chaotic behaviour.

1 Introduction

Since the discovery of unpredictability in deterministic systems that led to the study of chaotic dynamical systems, much work has been done to find properties of chaos that could give a classification of these systems. In particular indicators like Lyapounov exponent, Kolmogorov-Sinai entropy, topological entropy and others have been developed.

Nevertheless, in recent years there have been found dynamical systems for which all the known indicators don't give the presence of chaos, but

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numerical results show a high order of unpredictability for orbits of these systems. This phenomenon has led to attempts to generalize the known indicators (for example see [15],[16]), but has also stimulated the research for new properties of these systems.

One of the possible new approaches to these *weakly chaotic* dynamical systems is related to information theory, that is to the notion of information content of a symbolic string. In Section 2 we give the definition of information content and show how to apply this technique to dynamical systems.

In particular we study the dynamics of the logistic map (see equation (3)) at the chaos threshold, that represents one of the best known examples of weak chaos. In Section 3 we give the theoretical results that allow a classification of our map among the weakly chaotic dynamical systems.

Our theoretical approach is based on the notion of *Algorithmic Information Content* introduced by Kolmogorov and Chaitin (see Section 2), and unfortunately this notion of information is not computable, in the sense that cannot exist an algorithm which computes this function. Hence to numerically confirm our results, we have to use an approximate notion of information content. In Section 4, we present a possible approach to this problem, that consists on the use of compression algorithms to obtain an estimate for the information contained in a string. In particular a new compression algorithm, called CASToRe, is presented and some of its properties are illustrated. Finally in Section 5, we apply CASToRe to the logistic map at the chaos threshold to confirm the theoretical predictions for the behaviour of the Algorithmic Information Content of the strings generated by the map.

2 Information of symbolic sequences

In our approach the basic notion is the *information content of a symbolic sequence*. Given a finite string s^n of length n , that is a finite sequence of symbols s_i , $i = 1, \dots, n$, taken in a given alphabet, the intuitive meaning of *quantity of information* $I(s^n)$ contained in s^n is the following one:

$I(s^n)$ is the length of the smallest binary message from which you can reconstruct s^n .

Formally, this notion leads to the definition of the *Algorithmic Information Content (AIC)* (or *Kolmogorov complexity*) introduced by Chaitin ([6]) and Kolmogorov ([13]). In order to define it, it is necessary to define the

partial recursive functions. We limit ourselves to give an intuitive idea which is very close to the formal definition. We can consider a partial recursive function as a computer C which takes a program p (namely a binary string) as an input, performs some computations and gives a string $s = C(p)$, written in the given alphabet, as an output. The AIC of a string s is defined as the shortest binary program p which gives s as its output, that is

$$I_{AIC}(s, C) = \min\{|p| : C(p) = s\}, \quad (1)$$

where $|p|$ means the length of the string p . A computing machine is called *universal* (namely it is a computer in the usual sense of the word) if it can simulate any other machine (for a precise definition see any book on recursion). In particular if C and C' are universal then $I_{AIC}(s, C) \leq I_{AIC}(s, C') + const$, where the constant depends only on C and C' . This implies that, if C is universal, the complexity of s with respect to C depends only on s up to a fixed constant and then its asymptotic behavior does not depend on the choice of C . Thus, we will just write $I_{AIC}(s)$ in equation (1) when referring to the AIC of the string s .

We now extend the concept of information to strings generated by a dynamical system (X, f) . Using the usual procedure of symbolic dynamics, given a partition α of the phase space of the dynamical system (X, f) , it is possible to associate a string $\Phi_\alpha(x)$ to the orbit having $x \in X$ as initial condition. If $\alpha = \{A_1, \dots, A_l\}$, then $\Phi_\alpha(x) = (s_0, s_1, \dots, s_k, \dots)$ if and only if

$$f^k x \in A_{s_k} \quad \forall k,$$

with $s_k \in \{1, \dots, l\} \forall k$. The AIC of a single orbit of the dynamical system is then the AIC of the symbolic string generated by the orbit. If we want not to have dependence on the partition α of the phase space, then we have to make some procedure of looking for the supremum varying the partitions. The first results have been obtained by Brudno ([5]) using open covers of the phase space. Another possible approach, using computable partitions¹, is introduced in [3]. We are not interested in this problem, but we will just assume that the partitions we consider are *generating* in the sense that they give the best approximation for the AIC of an orbit of our dynamical system.

¹The intuitive idea of computable partition is a partition that can be recognised by a computer.

There exist some results connecting the information content of a string generated by a dynamical system and the Kolmogorov-Sinai entropy h^{KS} of the system.

First of all it is proved that in a dynamical system with an ergodic invariant measure μ with positive K-S entropy h_μ^{KS} , the AIC of a string n symbols long behaves like $I_{AIC}(s^n) \sim h_\mu^{KS} n$ for almost any initial condition with respect to the measure μ ([5]).

Instead, in a periodic dynamical system, we expect to find $I_{AIC}(s^n) = O(\log(n))$. Indeed, the shortest program that outputs the string s^n would contain only information on the period of the string and on its length.

It is possible to have also intermediate cases, in which the K-S entropy is null for all the invariant measures that are physically relevant and the system is not periodic. These systems, whose behaviour has been defined *weak chaos*, are an important challenge for research on dynamical systems. Indeed no information are given by the classical properties, such as K-S entropy or Lyapounov exponents, and in the last years some generalized definitions of entropy of a system have been introduced to characterize the behaviour of such systems (for example see [15]). We believe that an approach to weakly chaotic systems using the order of increasing of their AIC could be a powerful way to classify these systems. This approach has already been used for the Manneville map ([11], [4]):

$$f(x) = x + x^z \pmod{1} \quad x \in [0, 1] , \quad z \geq 1 .$$

In the cited papers it is proved that the AIC of the Manneville map behaves, for values $z > 2$ of its parameter, as

$$I_{AIC}(s^n) \sim n^\alpha, \quad \alpha = \frac{1}{z-1} < 1 , \quad (2)$$

and this behaviour has been defined *sporadicity*.

Among the weakly chaotic systems, one can identify behaviours different from the sporadic one. As an example, the AIC can have order smaller than any power law (with respect to the length of the encoded string). We choose to call this behaviour **mild chaos**.

In this paper we show that the AIC of the logistic map at the chaos threshold is of order of $\log n$ (Theorem 3.10 and Section 5). Then, we will prove that the weakly chaotic dynamics of the logistic map at the chaos threshold is in particular *mildly chaotic*.

3 The logistic map from information point of view

We now apply the theory of the Algorithmic Information Content of an orbit of a dynamical system to the logistic map at the chaos threshold. We first give some well known results on the dynamics of the map, and then use these results to obtain an estimate for the AIC.

3.1 The dynamics of the logistic map

The logistic map, defined by

$$f_\lambda(x) = \lambda x(1 - x) , \quad x \in [0, 1] , \quad 1 \leq \lambda \leq 4, \quad (3)$$

is a very simple example of a map with an extremely complicated behaviour for some values of the parameter λ . The dynamics of the map have been studied extensively, and there are many important results that have been generalized to one dimensional dynamical systems. Here we give a brief description of the well known period doubling sequence, and recall some results for the dynamics at the chaos threshold. The references for the first part of this section are [7], [8], [9].

Consider first $\lambda \in (1, 3)$. There are two fixed points, $x_0 = 0$ and $x_1 = (\lambda - 1)/\lambda$. The fixed point at the origin is unstable and the point x_1 is stable. Then every orbit ends up on the fixed point x_1 .

When $\lambda = \lambda_1 = 3$, we have the first bifurcation. The derivative of f at the point x_1 is -1, and the fixed point is now neutrally stable. Moreover one periodic orbit of period 2 is generated and it becomes stable as soon as λ becomes greater than λ_1 . This is a period doubling bifurcation.

This kind of bifurcation repeats over and over at different values λ_n of the parameter. That is, when $\lambda \in (\lambda_n, \lambda_{n+1})$, there is one stable periodic orbit of period 2^{n+1} , and unstable periodic orbits of period 2^j , $j = 0, \dots, n$. When $\lambda = \lambda_{n+1}$, the stable periodic orbit loses its stability and a new periodic orbit of period 2^{n+2} is generated.

One of the main features of this bifurcation sequence is that the bifurcation parameters λ_n accumulate at $\lambda = \lambda_\infty$, where

$$\lambda_\infty = 3.569945671870944901842 . \quad (4)$$

Moreover it holds

$$\lambda_n = \lambda_\infty - \frac{b}{\delta^n}, \quad (5)$$

where δ is the so-called *Feigenbaum constant* and b is a suitable constant ([10]).

The behaviour of the logistic map at the chaos threshold has attracted much attention, in particular for being a map with null Kolmogorov entropy for any invariant probability measure, and nevertheless showing a weakly chaotic behaviour.

We recall that in this paper by *weak chaos* we mean the behaviour of a map with null Kolmogorov entropy for all the physically relevant invariant measures and with a dynamics that is neither periodic nor regular in some sense. As we have seen in Section 2, a way to classify weakly chaotic maps is given by the *Algorithmic Information Content (AIC)* of a string generated by the map.

Let's consider now the map f_{λ_∞} . For this map there are countably many unstable periodic orbits of periods 2^j , for all $j \in \mathbb{N}$, and an attractor Ω that is a Cantor set, the so-called *Feigenbaum attractor*. It holds the following theorem ([7], Theorem III.3.5):

Theorem 3.1. *The logistic map f_{λ_∞} at the chaos threshold has an invariant Cantor set Ω .*

(1) *There is a decreasing chain of closed subsets*

$$J^{(0)} \supset J^{(1)} \supset J^{(2)} \supset \dots,$$

each of which contains $1/2$, and each of which is mapped onto itself by f_{λ_∞} .

(2) *Each $J^{(i)}$ is a disjoint union of 2^i closed intervals. $J^{(i+1)}$ is constructed by deleting an open subinterval from the middle of each of the intervals making up $J^{(i)}$.*

(3) *f_{λ_∞} maps each of the intervals making up $J^{(i)}$ onto another one; the induced action on the set of intervals is a cyclic permutation of order 2^i .*

(4) *$\Omega = \cap_i J^{(i)}$. f_{λ_∞} maps Ω onto itself in a one-to-one fashion. Every orbit in Ω is dense in Ω .*

(5) *For each $k \in \mathbb{N}$, f_{λ_∞} has exactly one periodic orbit of period 2^k . This periodic orbit is repelling and does not belong to $J^{(k+1)}$. Moreover this periodic orbit belongs to $J^{(k)} \setminus J^{(k+1)}$, and each point of the orbit belongs to one of the intervals of $J^{(k)}$.*

(6) Every orbit of f_{λ_∞} either lands after a finite number of steps exactly on one of the periodic orbits enumerated in 5, or converges to the Cantor set Ω in the sense that, for each k , it is eventually contained in $J^{(k)}$. There are only countably many orbits of the first type.

A characteristic of chaotic dynamical systems is the *sensitivity to initial conditions*. Roughly speaking we can say that a system has sensitive dependence on initial conditions if two orbits that start close diverge.

It is well known that the Kolmogorov-Sinai entropy is related to the sensitivity to initial conditions of the orbits. The exact relations between the K-S entropy and the instability of the system is given by the Ruelle-Pesin theorem. We will recall this theorem in the one-dimensional case. Suppose that the average rate of separation of nearby starting orbits is exponential, namely

$$\Delta x(n) \simeq \Delta x(0)2^{\lambda n} \quad \text{for } \Delta x(0) \ll 1,$$

where $\Delta x(n)$ denotes the distance of these two points at time n . If the Lyapounov exponent λ is positive then the system is unstable and λ can be considered a measure of its instability (or sensibility with respect to the initial conditions). The Ruelle-Pesin theorem implies that, under some regularity assumptions, λ equals the K-S entropy.

The logistic map at the chaos threshold has null Kolmogorov entropy for any invariant probability measure (it is shown by continuity of topological entropy and by the Variational Principle), and null Lyapounov exponent (experimental results), but nevertheless numerical experiments show that there is a power law divergence for nearby orbits. This feature of the logistic map has induced the application of a generalized version of the thermodynamical entropy ([15], [16]). Formally, we have:

Definition 3.2. A dynamical system $f : X \rightarrow X$ has *sensitivity to initial conditions* if there exists $\delta > 0$ such that, for all $x \in X$ and for all neighbourhoods U of x , there exist $y \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$.

Theorem 3.3. *The logistic map at the chaos threshold has no sensitivity to initial conditions. Indeed there exists a subset $X \subset [0, 1]$ with $m(X) = 1$, for the Lebesgue measure m , such that for all $\delta > 0$ and for all $x \in X$, there exists a neighbourhood $U(\delta)$ of x such that $\forall y \in U(\delta)$ and $\forall n \in \mathbb{N}$ we have $d(f_{\lambda_\infty}^n(x), f_{\lambda_\infty}^n(y)) < \delta$.*

Proof. By Theorem 3.1(6), we have that eventually almost every orbit, with respect to the Lebesgue measure m , ends up on $J^{(k)}$ for all k . Moreover by Theorem 3.1(1), we have that $f_{\lambda_\infty}(J^{(k)}) \subset J^{(k)}$.

Let now $\delta > 0$ be fixed. Being Ω a Cantor set, there exists k_δ such that $m(J_i^{(k_\delta)}) < \delta$ for all $i = 1, \dots, 2^{k_\delta}$, where by $J_i^{(k_\delta)}$ we denote the intervals making up $J^{(k_\delta)}$. We have (Theorem 3.1(3)) that the action of f_{λ_∞} on $J^{(k_\delta)}$ is a cyclic permutation of the 2^{k_δ} intervals. Then if $x \in J_i^{(k_\delta)}$, for some i , and $U(\delta)$ is a neighbourhood of x contained in $J_i^{(k_\delta)}$, then, for all n , $f_{\lambda_\infty}^n(U(\delta)) \subset J_j^{(k_\delta)}$, for some j . Hence $\forall y \in U(\delta)$ and $\forall n \in \mathbb{N}$, we have $d(f_{\lambda_\infty}^n(x), f_{\lambda_\infty}^n(y)) < \delta$.

Let now X be the set of the points whose orbit is eventually contained in $J^{(k_\delta)}$. By Theorem 3.1(6), we have $m(X) = 1$. Let now be $x \in X$, then there exists a neighbourhood $U(\delta)$ of x , such that, if $f_{\lambda_\infty}^N(x) \in J_i^{(k_\delta)}$ for some i , then $f_{\lambda_\infty}^N(U(\delta)) \subset J_i^{(k_\delta)}$. The neighbourhood $U(\delta)$ exists by continuity of the map f_{λ_∞} . Now, we can repeat the same argument as before, showing that $\forall y \in U(\delta)$ and $\forall n \in \mathbb{N}$, we have $d(f_{\lambda_\infty}^n(x), f_{\lambda_\infty}^n(y)) < \delta$ (if necessary it is possible to shrink the neighbourhood $U(\delta)$ to have the previous relation for all $n \in \mathbb{N}$). \square

Hence it remains the problem of explaining the power law divergence of nearby orbits. One way to give an estimate of the order of this divergence is by considering the dimension of the set of points that remain close for some iterations.

Definition 3.4. For $x \in [0, 1]$, we define

$$B(x, n, \epsilon) = \{y \in [0, 1] \mid d(f_{\lambda_\infty}^i(x), f_{\lambda_\infty}^i(y)) < \epsilon \ \forall i = 0, \dots, n\}.$$

Using this definition, we apply Theorem 3.3 to obtain

Corollary 3.5. For all $\epsilon > 0$ and for all $x \in X$, there exists a $N(x, \epsilon) \in \mathbb{N}$, such that $B(x, n, \epsilon) = B(x, m, \epsilon)$ for all $n, m \geq N(x, \epsilon)$.

Hence, given a point $x \in X$ and ϵ , the set $B(x, n, \epsilon)$ shrinks as n increases, until n reaches the value $N(x, \epsilon)$. At this point the set $B(x, n, \epsilon)$ doesn't change any more. But, if we are interested in estimate the order of divergence of nearby orbits at the point x , we have to consider the limit

$$B(x, n) = \lim_{\epsilon \rightarrow 0^+} B(x, n, \epsilon). \tag{6}$$

We find that $\lim_{\epsilon \rightarrow 0^+} N(x, \epsilon) = +\infty$, for all $x \in X$, hence the function $B(x, n)$ is a decreasing function of n , whose order gives information on the order of the local divergence. In the next subsection we derive an estimate for the AIC of a string generated by f_{λ_∞} , and hence, by a theorem of [12], we find the order of $B(x, n)$.

3.2 The AIC at the chaos threshold

In this subsection we show an approach to the dynamics of the logistic map at the chaos threshold, f_{λ_∞} , that allows us to find an estimate for the AIC (equation (1)) of almost every orbit generated by f_{λ_∞} . This approach uses the notion of *kneading invariant*. To have a complete treatment of *kneading theory* see, for example, [8].

Let $f : [0, 1] \rightarrow [0, 1]$ be a C^1 unimodal map, that is there exists only one point $c \in [0, 1]$ such that $f'(c) = 0$, and f is increasing on $(0, c)$ and decreasing on $(c, 1)$. We can then define the *kneading sequence* $k_f(x, t)$ of a point $x \in [0, 1]$.

Definition 3.6. For $x \in [0, 1]$, we define coefficients $\theta_i(x)$ by:

$$\theta_i(x) = \begin{cases} +1 & \text{if } \frac{d}{dx} f^{(i+1)} > 0 \\ -1 & \text{if } \frac{d}{dx} f^{(i+1)} < 0 \\ 0 & \text{if } \frac{d}{dx} f^{(i+1)} = 0 \end{cases}$$

Then the *kneading sequence* relative to x is the formal power series

$$k_f(x, t) = \sum_{i=0}^{\infty} \theta_i(x) t^i.$$

Theorem 3.7. *It is possible to introduce a distance D on the space of kneading sequences, given by*

$$D(k_f(x, t), k_f(y, t)) = \sum_{i=0}^{\infty} \frac{1}{2^i} |\theta_i(x) - \theta_i(y)|.$$

This distance makes the function $x \rightarrow k_f(x, t)$ a continuous function for all $x \in [0, 1]$, but the preimages of the critical point c .

Definition 3.8. If x is a preimage of the critical point c , we define

$$k_f(x_{\pm}, t) = \lim_{y \rightarrow x^{\pm}} k_f(y, t),$$

where the limit is taken through sequences of points y that are not preimages of c . For the critical point c , it holds $k_f(c_-, t) = -k_f(c_+, t)$. Then we define the *kneading invariant* $k_f(t)$ of the function f as the sequence $k_f(c_-, t)$.

Using classical results on renormalization of maps with null topological entropy ([8]), one can prove the following:

Theorem 3.9. *The kneading invariant of the logistic map at the chaos threshold f_{λ_∞} is given by*

$$k_{f_{\lambda_\infty}}(t) = \prod_{i=0}^{\infty} (1 - t^{2^i}). \quad (7)$$

Let's consider now the partition $\alpha = (A_0, A_1)$, given by $A_0 = [0, 1/2]$ and $A_1 = (1/2, 1]$, of the interval $[0, 1]$ for the map f_{λ_∞} . If we take a point $x \in [0, 1]$, we can code its orbit into a string s with $s_i \in \{0, 1\}$, according to whether $f_{\lambda_\infty}^i(x)$ is in A_0 or in A_1 .

From the definition of the coefficients $\theta_i(x)$, for any point x , we can see that $\theta_i(x) = \text{sgn } (f'(x) \cdots f'(f^i(x)))$. Hence, in the coding of the orbit of x into the string s , we have $s_i = 0$ if there is a permanence in the sign of $\theta_{i-1}(x)$ and $\theta_i(x)$, $s_i = 1$ otherwise.

We denote by $s(x)$ the string generated by the orbit with initial condition x . Let's start studying the string generated by the point $x = 1/2$, that is the critical point of f_{λ_∞} . The coding of the orbit of $1/2$ into a string s is related to the kneading invariant of the map $k_{f_{\lambda_\infty}}(t)$, given by Theorem 3.9. Thanks to the particular form of the kneading invariant (eqn. (7)), we obtain that to reconstruct the string s , it is sufficient to make the following operations:

- the first symbol is 0;
- given s^n , that is the first n symbols of the string, we construct s^{2n} by $s^n \bar{s}^n$, where \bar{s}^n is equal to s^n if $n = 2^k$, otherwise \bar{s}^n is obtained by s^n changing the first symbol.

Hence we have that $I_{AIC}(s^n(1/2)) = \log n$, indeed it is enough to specify the length of the string s^n . Moreover, by Theorem 3.1(1), we have that

$1/2 \in \Omega$, the attractor of f_{λ_∞} , and the orbit of $1/2$ is dense in Ω (Theorem 3.1(4)). Hence for all $x \in X$ (see Theorem 3.3), there exists $N(x)$ such that for all $i \geq N(x)$, $s_i(x)$ can be obtained by $s_i(1/2)$. This implies the following result:

Theorem 3.10. *For almost any point $x \in [0, 1]$, the AIC of the string $s(x)$ generated by the orbit of the map f_{λ_∞} with x as initial condition, is such that*

$$I_{AIC}(s^n(x)) = \log n + \text{const}, \quad (8)$$

where the constant depends on the point x .

Remark 3.11. We remark that we have actually proved only the behaviour of the AIC for the partition α . Indeed to obtain the AIC of the orbits of our dynamical system we should consider the supremum on either all the open covers or the computable partitions (see Section 2). But we have that α is a *generating* partition, that is $\lim_{n \rightarrow \infty} \text{diam}(\alpha^n) = 0$, where $\text{diam}(\alpha)$ is the diameter of a partition, and $\alpha^n = \alpha \wedge \dots \wedge f^{-n}(\alpha)$. Hence we suppose that the AIC we estimated is a good approximation of the real one, following the same arguments used, for example, for the Kolmogorov-Sinai entropy.

We are now ready to give an estimate on the behaviour of the function $B(x, n)$ defined in equation (6). Indeed, using Theorem 40 in [12], we have:

Corollary 3.12. For almost any point $x \in [0, 1]$, the function $B(x, n)$ is of the order of n^{-k} , for some constant $k > 0$. Hence, we can say that we have a power law divergence for nearby orbits, for almost any orbit.

This result confirms the experiments made on the logistic map at the chaos threshold. The constant k has been found to be approximately 1.3236 ([16]).

We have thus derived an estimate for the behaviour of the Algorithmic Information Content for the logistic map at the chaos threshold, showing its connection with the sensitivity to initial conditions. We remark that the AIC is not a computable function, that is it does not exist an algorithm able to compute the AIC of any string. In the next section, we show how to obtain an estimate of the AIC for our map.

4 Compression Algorithms

In Section 2 we have introduced the notion of Algorithmic Information Content (AIC) of a finite string s , following the work of Chaitin ([6]) and Kolmogorov ([13]). Since the notion of AIC is not computable, it has been approximated by other notions of information content of a string that can be computed.

One measure of the information content of a finite string built on an alphabet \mathcal{A} can be defined by a lossless (reversible) data compression algorithm

$$Z : \Sigma(\mathcal{A}) \rightarrow \Sigma(\{0, 1\}) ,$$

that is a coding procedure such that from the coded string we can reconstruct the original string. Hence it is natural to consider the length of the coded string as an approximate measure of the quantity of information that is contained in the original string. We then define the information content $I_Z(s)$ of the string s as

$$I_Z(s) = |Z(s)| , \quad (9)$$

where $|Z(s)|$ denotes the length of the compressed string $Z(s)$. The information content $I_Z(s)$ turns out to be a computable function and for this reason we will call it Computable Information Content (*CIC*). For a more accurate discussion on compression algorithms and the CIC we refer to [3].

4.1 A new compression algorithm: CASToRe

In [1] and [3], it has been presented a new compression algorithm, called CASToRe, which has been created in order to give information on null entropy dynamics and has been used to study a case of sporadic dynamics, the Manneville map. In the following, the algorithm CASToRe will be used to evaluate the Computable Information Content I_Z of the strings (equation (9)). As will be better shown later in the case of regular strings, this new algorithm will be a sensitive measure of the information content of the string. That's why it is called **CASToRe**: **C**ompression **A**lgorithm, **S**ensitive **T**o **R**egularity. In particular, in [3] the algorithm CASToRe has been tested on fully chaotic and sporadic strings, confirming the theoretical results.

We now give a short description of the algorithm. CASToRe is a compression algorithm which is a modification of the LZ78 algorithm ([14]).

As it will be proved in Theorem (4.1), the Information I_Z of a constant sequence, originally with length n , is $4 + 2 \log(n+1)[\log(\log(n+1)) - 1]$, if the algorithm Z is CASToRe. The theory predicts that the best possible information is $I_{AIC} = \log(n) + \text{const}$. In [1], it is shown that the algorithm LZ78 encodes a constant n digits long sequence to a string with length about $\text{const} + n^{\frac{1}{2}}$ bits; so, we cannot expect that LZ78 is able to distinguish a sequence whose information grows like n^α ($\alpha < \frac{1}{2}$) (sporadic dynamics) from a constant or periodic one, while it has been proved ([3]) that CASToRe is a very useful tool to identify the correct exponent in the sporadic case.

The algorithm CASToRe is based on an adaptive dictionary. Roughly speaking, this means that it translates an input stream of symbols (the file we want to compress) into an output stream of numbers, and that it is possible to reconstruct the input stream knowing the correspondence between output and input symbols. This unique correspondence between sequences of symbols (words) and numbers is called *the dictionary*.

At the beginning of encoding procedure, the dictionary is empty. In order to explain the principle of encoding, let's consider a point within the encoding process, when the dictionary already contains some words.

We start analyzing the stream, looking for the longest word W in the dictionary matching the stream. Then we look for the longest word Y in the dictionary where $W + Y$ matches the stream. Suppose that we are compressing an English text, and the stream contains “basketball ...”, we may have the words “basket” (number 119) and “ball” (number 12) already in the dictionary, and they would of course match the stream.

The output from this algorithm is a sequence of word-word pairs (W, Y) , or better their numbers in the dictionary, in our case (119, 12). The resulting word “basketball” is then added to the dictionary, so each time a pair is output to the code-stream, the string from the dictionary corresponding to W is extended with the word Y and the resulting string is added to the dictionary.

In the following subsections, we will estimate the information content I_Z , where the algorithm Z is CASToRe, in the case of constant sequences and periodic sequences with period $p > 1$. Indeed this is important for the application of CASToRe to the logistic map at the chaos threshold (Section 5).

4.2 CASToRe on constant strings

We first study the case of period $p = 1$ (constant strings). If we have a string given by

$$s = (aaaaaaaaaaaaaaa.....) , \quad (10)$$

the algorithm CASToRe gives a codification

$$\begin{aligned} 1 &: (0, a) & [a] \\ 2 &: (1, 1) & [aa] \\ 3 &: (2, 2) & [aaaa] \\ 4 &: (3, 3) & [aaaaaaaa] \\ &\dots \\ k &: (k-1, k-1) & [a\dots a] \end{aligned} \quad (11)$$

so if we compute the information $I_Z(s^n)$, as function of the length n of the string to be compressed, we can immediately deduce that $I_Z(s^n) \sim O(\log_2 n)$. Indeed a much more accurate estimate of the CIC for constant strings is obtained in the following theorem.

Theorem 4.1. *Let s be a constant string as in (10), then the information function $I_Z(s^n)$ obtained applying the compression algorithm CASToRe is approximated by the function $\Psi(n)$:*

$$\Psi(n) = 4 + 2 \log_2(n+1) (\log_2(\log_2(n+1)) - 1) , \quad (12)$$

where the approximation is given by the identification of the integer part of $\log_2(\log_2(n))$ with its real value.

Proof. We notice that the algorithm CASToRe acts on s in such a way that at the end of each term of the codification, the length n of the codified string is of the form $n = 2^h - 1$. Then we look for the value of $I(2^h - 1)$. We can easily obtain that if $k = [\log_2 h]$, where $[\cdot]$ denotes the integer part, then

$$\begin{aligned} I(2^h - 1) &= 2 + \sum_{i=0}^{k-1} 2(2^{i+1} - 2^i) \log_2(2^{i+1}) + 2(n - 2^k) \log_2(2^{k+1}) \\ &= 2 + \sum_{i=0}^{k-1} (i+1)2^{i+1} + 2(n - 2^k) \log_2(2^{k+1}) \\ &= 4 + (k-1)2^{k+1} + 2(n - 2^k) \log_2(2^{k+1}), \end{aligned} \quad (13)$$

where we used

$$\sum_{j=1}^r j2^j = (r-1)2^{r+1} + 2.$$

At this point it is enough to approximate k with its real value $\log_2 h$, and substitute to h its value $\log_2(n+1)$ to obtain equation (12). The theorem is proved. \square

4.3 CASToRe on periodic strings

We now look at strings s with prime period $p > 1$. From the point of view of the information content, we expect that periodic strings are not so different from constant strings. Indeed we find that the algorithm CASToRe is able to recognize periodic strings as constant strings after having encoded n_p symbols, depending on p . Then, the CIC $I_Z(s^n)$ must behave as in the constant case for $n > n_p$, indeed it is given by $\Psi(n)$ plus a constant term C_s that depends on the particular string s . This term C_s can be estimated as the amount of information obtained from s at the n_p -th symbol. We first prove the following theorem

Theorem 4.2. *Let s be a periodic string with prime period $p > 1$, then there exists a constant n_p , depending on s , such that, when the algorithm CASToRe reaches the n_p -th term of the string, it starts codifying the string s as if it were a constant string. The constant n_p has as upper bound*

$$K_p = p^2 \left[2 + \frac{3}{2} \log_2 p + \frac{1}{2} \log_2 |\mathcal{A}| \right] \quad (14)$$

where $|\mathcal{A}|$ is the number of symbols in the alphabet used to construct the string s . Note that K_p depends only on the period p .

Proof. Let us assume that $|\mathcal{A}| = r$. Since s has prime period p , we can look at it as a sequence of substrings, each one p symbols long; they will be called p -substrings.

Once the codification has started and the first p characters have been read, the last encoded word will be t symbols long and will end at some k -th (mod. p) site in the second p -substring. Then, after at most $(p+1)$ subsequent reading of contiguous p -substrings, the algorithm will arrive at

the end of an already known word; this word will finish at some h -th (mod. p) site of the $(p + 2)$ -th p -substring of s .

Let us assume that the number of words needed by the algorithm to codify the fragment starting from the h -th symbol of a previous p -substring to its correspondent belonging to the $(p + 2)$ -th p -substring of s is m ; we will indicate the collection of that m words by $\mathcal{F}(h)$. Let us analyse the different cases.

First case: $m = 2^c$. The algorithm continues to process the remaining part of s , codifying the next fragment $\mathcal{F}(h)$ by coupling two words each time. So, after c steps only *one* word will be necessary and later the algorithm will follow the period.

Second case: $m = 2^{c_1}q$, where q is odd. The algorithm first behaves c_1 times as described above, and after just q words will be needed to process the following $\mathcal{F}(h)$. Let us call A_1, \dots, A_q those known words. After one step, the algorithm CASToRe will have built the new words $B_1 = (A_1, A_2), \dots, B_{\frac{q-1}{2}} = (A_{q-2}, A_{q-1})$, and $B_{\frac{q-1}{2}+1} = (A_q, B_1)$. Then, the algorithm will reach the period after c_2 steps, where $c_2 = \min\{c \in \mathbb{N} : 2^c > q + 1\}$.

Third case: m is odd. The algorithm will behave as in the second case, with $c_1 = 0$.

So we have proved the existence of the constant n_p . Let us estimate its upper bound, K_p . It holds:

$$n_p \leq p(p + 2)[1 + C]$$

where $C = \min\{c \in \mathbb{N} : 2^c > m + 1\}$. Now, we estimate m . It does hold $m \leq (p + 2) \times S$ where S is the number of words necessary to cover up a p -substring. Since r different symbols are available in the alphabet \mathcal{A} , then it must be $S \leq N$ where N is the smaller integer such that it holds $p \leq \sum_{i=1}^N ir^i$.

Since we have

$$p \leq \frac{r - (N + 1)r^{N+1} + Nr^{N+2}}{(r - 1)^2} = \sum_{i=1}^N ir^i$$

and we are interested in an estimation of N , let us look for the smallest integer N such that

$$\log_r \left(\frac{p(r - 1)^2}{r} - 1 \right) \leq 2 \log_r N.$$

Let N_0 the real solution of the following equation:

$$N_0 = \sqrt{\frac{(r-1)^2 p}{r} - 1}.$$

Then the integer N we looked for is $O(\sqrt{rp})$ and it holds $m \leq (p+2)\sqrt{rp}$. So, we have $C \leq 1 + \log_2(m+1)$. Finally, if we approximate $p+2$ with p , we obtain

$$K_p \leq p^2 \left[2 + \frac{3}{2} \log_2 p + \frac{1}{2} \log_2 r \right]$$

and the theorem is proved. \square

5 Experimental results on the logistic map

In this section we show how to obtain an estimation for the Computable Information Content $I_Z(s^n)$ for a finite string s^n generated by the logistic map at the chaos threshold, using CASToRe as compression algorithm Z . In the remainder of this section we drop the subscript Z from the CIC, since we are always referring to CASToRe.

The plan of our experiments on the logistic map is to apply CASToRe to strings generated by the logistic map f_λ for different values of λ that approximate the chaos threshold λ_∞ . In particular we consider two approximating sequences of values of λ : the sequence of values $\lambda_j < \lambda_\infty$ at which the period doubling bifurcations occur (Section 5.1) and the sequence of values $\mu_j > \lambda_\infty$ at which the inverse tangent bifurcations occur (Section 5.2). Thus we obtain approximations for the CIC at the chaos threshold from below and from above, and the two coincide. We can then say to have obtained an approximation for the CIC computed with CASToRe for the logistic map at the chaos threshold f_{λ_∞} .

We remark that the symbolic strings are obtained from the orbits of the logistic map at the chaos threshold considering the partition $\alpha = (A_0, A_1)$ of $[0, 1]$ given by $A_0 = [0, 1/2]$ and $A_1 = (1/2, 1]$. The choice of this partition for the experiments is justified by its optimality as remarked in Section 3.2.

5.1 The period doubling sequence

In Section 4.3 we have analysed the behaviour of our algorithm CASToRe on periodic strings, obtaining that the CIC $I(s)$ of a periodic string s behaves

with respect to the length n of the encoded string as

$$I(s^n) = \Psi(n) + C_s, \quad (15)$$

where $\Psi(n)$ is given by equation (12) and C_s is a constant depending on the string s (Theorem 4.2).

We start considering the logistic map for values of the parameter λ in the sequence $(\lambda_j)_{j \in \mathbb{N}}$ given by the period doubling bifurcations, for which values the logistic map is periodic. The sequence $(\lambda_j)_{j \in \mathbb{N}}$ converges to the chaos threshold value λ_∞ from below and can be generated by the equation (5) (Section 3.1).

We generated the terms of the sequence $(\lambda_j)_{j \in \mathbb{N}}$ (where the differences are given up to 38 decimal digits) and iterated the logistic map 10^7 times. So we had strings of 10^7 symbols. When studying the CIC $I_j(s^n)$ for each λ_j , where s^n is the symbolic orbit of the logistic map with parameter $\lambda = \lambda_j$, we first tried to obtain the constant terms C_j , as expected from equation (15). We have obtained an increasing sequence of values C_j (for j that tends to infinity), and our results about the compression of periodic strings by the algorithm CASToRe (see Theorem 4.2) are supported by the computations of the following limits

$$\lim_{n \rightarrow \infty} \frac{I_j(s^n) - C_j}{\Psi(n)} = 1 \quad \forall j \in \mathbb{N} \quad (16)$$

As expected, equation (16) tells us that, after the compression of a long enough substring of our string s , we obtain the same information function as for constant strings, simply translated by a constant. If we denote by n_j the number of symbols that the algorithm has to process before reaching that behaviour, we have $n_j \rightarrow \infty$ as $j \rightarrow \infty$. So we have that $C_j \rightarrow \infty$ as $j \rightarrow \infty$.

The approximation of the information content $I_j(s^n)$ with a function $\Psi(n)$ of the form given by (12) plus a constant term is not very good for periodic strings with high period. This is because of the poorness of the upper bound of our computations: indeed, we know that this approximation is accurate for values of $n > n_j$ and there are strings whose n_j is bigger than 10^7 , the length of our orbits. In our case the period p_j of the strings s_j is already big enough for $j = 12$, indeed $p_{12} = 2^{12} = 4098$, so $n_{12} \approx 16 \times 10^6$. This means that we have to look for another way of approximation to the information function for values of n lower than n_j . At the bifurcation points it is well known from experimental results that the orbit separation is polynomial and, thanks to

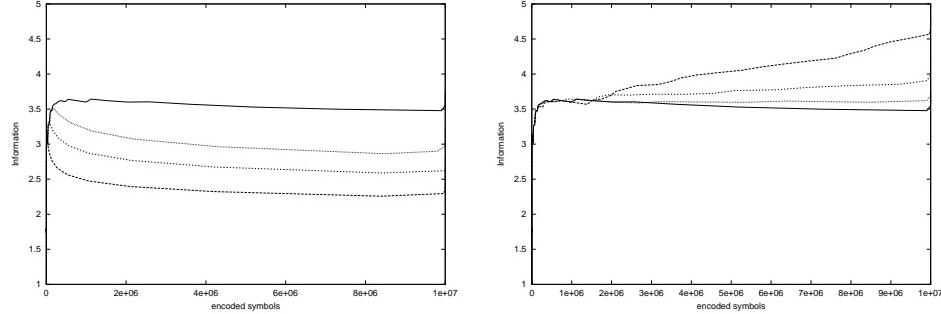


Figure 1: On the left one can see the behaviour of some of the functions $S_j(n)$ (dashed lines), that shows the monotonicity of these functions. The solid line is the function $S_\infty(n)$ and one can see that it is an upper bound for the functions $S_j(n)$. On the right there is the same picture but for the functions $S_k(n)$ that approximate $S_\infty(n)$ from above.

the results in [12], we can say that the Algorithmic Information Content is at most logarithmic. So, we can expect our Computable Information Content to behave like

$$I_j(s^n) \sim \begin{cases} S_j(n)\Psi(n) & \text{for } n < n_j \\ C_j + \Psi(n) & \text{for } n > n_j \end{cases} \quad j \in \mathbb{N} \quad (17)$$

with C_j as given before. To find the functions $S_j(n)$, we simply have to compute the fraction

$$S_j(n) = \frac{I_j(s^n)}{\Psi(n)}, \quad j \in \mathbb{N}. \quad (18)$$

First of all, we deduce from equation (16) that $\lim_{n \rightarrow \infty} S_j(n) = 1$ for all j . Moreover the numerical experiments show that the sequence of functions $(S_j(n))_{j \in \mathbb{N}}$ is point-wise increasing in j and bounded from above (see figure 1 on the left). Then we define the bounding function $S_\infty(n)$ as

$$S_\infty(n) = \lim_{j \rightarrow \infty} \left(S_j(n) \chi_{[0, n_j]}(n) + \chi_{[n_j, \infty]}(n) \right)$$

for each $n \in \mathbb{N}$, where $\chi_{[a, b]}(n)$ denotes the characteristic function of the real interval $[a, b]$.

5.2 Inverse tangent bifurcations

In order to establish the behaviour of the information function $I_\infty(s^n)$ at the chaos threshold, we applied the same method as before, building up a sequence $(\mu_k)_{k \in \mathbb{N}}$ of parameters approximating λ_∞ from above.

In the interval $[\lambda_\infty, 4]$, the logistic map has a general chaotic behaviour, except from narrow ranges (called *periodic windows*) of parameters for which the map is periodic. Inside each window, a new period doubling sequence can be identified, that leads the map to chaos. The behaviour becomes periodic from chaotic via a *tangent bifurcation*, that is we can find a fixed point of a given iterate of the map having 1 as its eigenvalue ([2]). The values of the parameter at which the tangent bifurcations occur converge to λ_∞ from above and can be generated using the following relation:

$$\mu_k = \lambda_\infty + \frac{c}{\delta^k}$$

where c is a suitable constant and δ is the Feigenbaum constant.

Using the same definition of $S_k(n)$ given in equation (18), we find a sequence of functions such that $\lim_{n \rightarrow \infty} S_k(n) = +\infty$ for all $k \in \mathbb{N}$, due to the chaotic behaviour of the map at μ_k .

As we can see in figure 1 on the right, the numerical evidence is that the sequence $(S_k(n))_{k \in \mathbb{N}}$ is point-wise decreasing in k and the same function $S_\infty(n)$ found in the previous subsection is a lower bound for the sequence. Then we have

$$\lim_{k \rightarrow \infty} S_k(n) = S_\infty(n)$$

for each n .

5.3 Results

From the previous subsections we can draw some conclusions. First of all, we can say that for all $n \in \mathbb{N}$ the information function $I_j(s^n)$ of the logistic map (3) with parameter $\lambda = \lambda_j$ (as shown in Section 5.1) is

$$I_j(s^n) \sim S_j(n)\Psi(n)\chi_{[0, n_j]}(n) + (\Psi(n) + C_j)\chi_{[n_j, \infty]}(n) \quad \forall j \in \mathbb{N},$$

and with parameter $\lambda = \mu_k$ (as shown in subsection 5.2) is

$$I_k(s^n) \sim S_k(n)\Psi(n) \quad \forall k \in \mathbb{N}.$$

Moreover we have that

$$\lim_{j \rightarrow \infty} I_j(n) = \lim_{k \rightarrow \infty} I_k(n) = I_\infty(n) = S_\infty(n)\Psi(n) \quad \forall n \in \mathbb{N}.$$

From the numerical approximation of $S_\infty(n)$ from above and below, we conjecture that $S_\infty(n)$ is an increasing bounded function of n and there exists a constant $S_\infty = \lim_{n \rightarrow \infty} S_\infty(n)$.

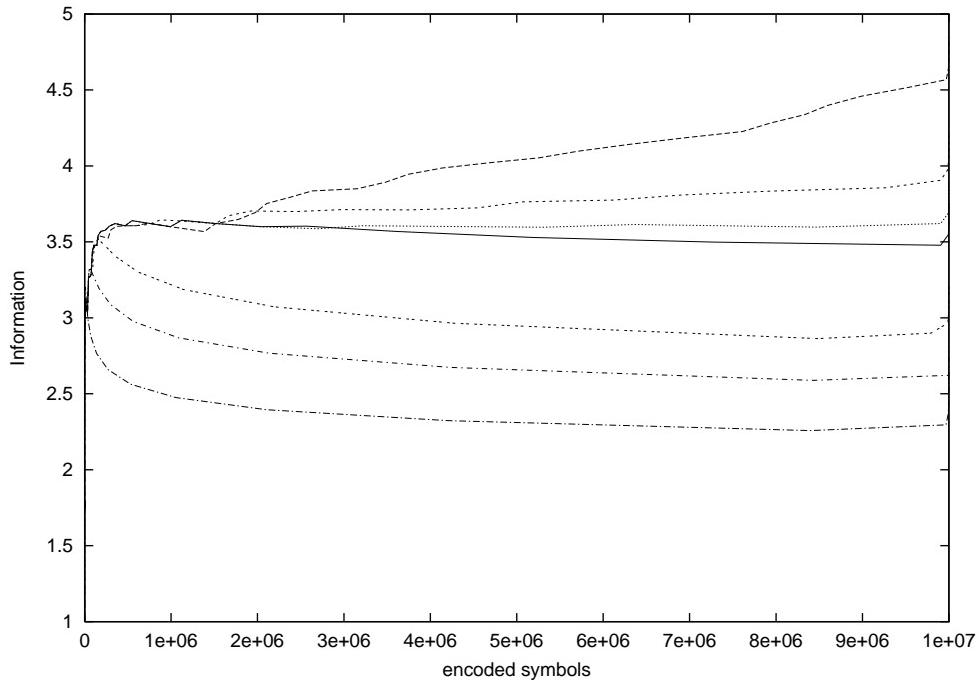


Figure 2: The solid line is the limit function $S_\infty(n)$, and dashed lines are the approximating functions $S_i(n)$ from above and below.

Numerical estimates give a value of S_∞ of more or less 3.5. In figure 2 the behaviour of functions $S_i(n)$ corresponding to some values of both the approximating sequences is shown.

We have thus confirmed the theoretical result of Theorem 3.10, proving that the logistic map at the chaos threshold is *mildly chaotic* (see Section 2).

Finally we remark that the feature of $I_\infty(s^n)$ and in particular of $S_\infty(n)$ is typical of the logistic map, and is not simply due to the fact that we are considering periodic orbits with period going to infinity. Indeed, from the

behaviour of the function $S_\infty(n)$, we deduce that $\max_n S_\infty(n) < 3.7$ and this means that for $n = 10^3$, for example, $I_\infty(s^n)$ is less than approximately 148. But we can construct periodic strings with period less than 100, such that the information after 10^3 symbols is more than 148. We can consider, for example, a string with period given by all the possible combinations of two symbols in pieces of length from 1 to 4. This string has period $98 \ll 2^7$, but $I(734) = 356$, much more than 148.

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